Response of an Euler-Bernoulli Beam Resting on a Pasternak Foundation Subjected to Harmonic High-Speed Moving Loading

*Askar Ibrayev[1], Desmond Adair[2], Jong R. Kim[3]

[1,3] Department of Civil and Environmental Engineering, Nazarbayev University, Nur-Sultan 010000, Kazakhstan
[2] Department of Mechanical and Aerospace Engineering, Nazarbayev University, Nur-Sultan 010000, Kazakhstan
[2] dadair@nu.edu.kz

ABSTRACT

An analytical solution technique that is fast and efficient and provides accurate results is developed for the fourth-order non-linear differential equation describing the transverse vibrations of a beam on a two parameter elastic foundation. The solution technique developed is based on a recent novel method of solution, the Adomian Modified Decomposition Method (AMDM) which allows solving without discretization, perturbation, linearization, or a priori assumptions. All of these have the potential to change the physics of the problem. Numerical calculations of vibration frequencies are performed and the effect of foundation parameters and loadings on beam vibrations are analyzed and discussed.

1. INTRODUCTION

High-speed railway transportation is gaining broad interest around the world due to their efficiency, transportation speed, and less pollution potential. However, one of the disadvantages of high-speed railways is track vibration and induced noise. Excessive track vibration can increase maintenance and down time of the track, require the decrease of train maximum allowable speed, reduce train service lifetime and also decrease the riding comfort of passengers. Therefore, the vibration response of the railway track to moving loading is important in the area of high-speed railway transportation. Theoretical model is developed using simple geometries of rail and supporting subsoil. It is assumed that the rails act as a finite beam resting on an elastic foundation, which is modelled using springs. This model allows developing mathematical formulations using the existing beam and foundation theories.

The method of solution developed here is based on a recent novel method of calculation, namely, the Adomian Modified Decomposition Method (AMDM) (Adomian 1994). This method has been used in other applications to provide fast and accurate results (Wazwaz 1999; Wazwaz and El-Sayed 2001). Compared to other existing methods the AMDM has advantages of computational simplicity and solving without

---

[1]Graduate Student
[2]Associate Professor
[3]Professor

discretization, perturbation, linearization, or a priori assumptions, all of which has the potential to change the physics of the problem (De Rosa and Maurizi 1998).

2. MATHEMATICAL FORMULATION

2.1 Governing equation

Transverse motion of an Euler-Bernoulli beam resting on an elastic two-parameter Pasternak foundation is illustrated in Fig. 1 and can be expressed as in Eq. (1) (Taha 2017)

$$EI\frac{\partial^4 w(x,t)}{\partial x^4} - k_p\frac{\partial^2 w}{\partial t^2} - \rho I\frac{\partial^4 w(x,t)}{\partial x^2 \partial t^2} + k_w w(x,t) + \rho A\frac{\partial^2 w(x,t)}{\partial t^2} = q(x,t) \tag{1}$$

where $q(x,t)$ is lateral excitation, k_w and k_p are the linear and shear stiffness of foundation, respectively.

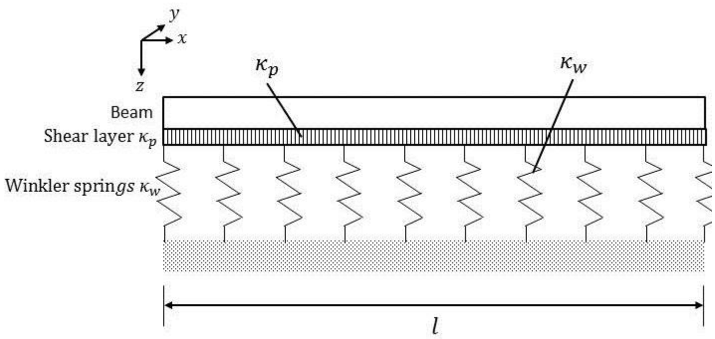

Fig. 1 An Euler-Bernoulli beam on a Pasternak foundation.

By using the modal analysis for harmonic free vibration, $w(x,t)$ is separated in space and time (Tazabekova et al. 2018; Adair et al. 2019; Adair et al. 2019) and Eq. (1) reduces to

$$EI\frac{\mathrm{d}^4\phi(x)}{\mathrm{d}x^4} + k_p\omega^2\phi(x) + \rho I\omega^2\frac{\mathrm{d}^2\phi(x)}{\mathrm{d}x^2} + k_w\phi(x) - \rho A\omega^2\phi(x) = q(x) \tag{2}$$

Eq. (2) is now made non-dimensional using

$$X = \frac{x}{l},\ \ \phi(X) = \frac{\phi(x)}{l},\qquad K_0 = \frac{k_w l^4}{EI},\qquad K_1 = \frac{k_p l^2}{EI}\quad \lambda = \frac{\rho A\omega^2 l^4}{EI},\qquad \gamma = l^2\frac{A}{I}, Q = \frac{qL^3}{EI}$$

and becomes

$$\frac{\mathrm{d}^4\phi(X)}{\mathrm{d}X^4} + \left(\frac{\lambda}{\gamma} - K_1\right)\frac{\mathrm{d}^2\phi(X)}{\mathrm{d}x^2} + (K_0 - \lambda)\phi(X) = Q(X) \tag{3}$$

2.2 Distributed harmonic high-speed moving loading

To apply the loading to the beam the model of a train consisting of a few wagons is considered. Loading generated by wagons can be expressed by the finite sum of harmonically varying loads and formulated as in Eq. (4) using the Heaviside function,

$$q(x,t) = \sum_{d=0}^{D-1} \frac{P_0}{2r} cos^2\left(\frac{\pi(x - vt - (2r+s)d)}{2r}\right) * H(r^2 - (x - vt - (2r+s)d)^2)e^{i\Omega t} \tag{4}$$

where P_0 is the load, $2r$ is the span of the load, v the velocity of moving load, $H(:)$ is the Heaviside function, d is a number of separated impulses, $\Omega = 2\pi f_\Omega$ frequency of the moving load and s is the distance between them.

Eq. (4) is made non-dimensional using

$$X = \frac{x}{l}, R = \frac{r}{l}, V = \frac{v}{l}, S = \frac{s}{l}$$

and becomes

$$Q(X,t) = \sum_{d=0}^{D-1} \frac{P_0}{2R} cos^2\left(\frac{\pi(X - Vt - (2R+S)d)}{2r}\right) * H(R^2 - (X - Vt - (2R+S)d)^2)e^{i\Omega t} \tag{5}$$

By substituting Eq.(5) to the r.h.s of the Eq. (3) one can obtain

$$\frac{\mathrm{d}^4\phi(X)}{\mathrm{d}X^4} + \left(\frac{\lambda}{\gamma} - K_1\right)\frac{\mathrm{d}^2\phi(X)}{\mathrm{d}x^2} + (K_0 - \lambda)\phi(X) = \sum_{d=0}^{D-1} \frac{P_0}{2R} cos^2\left(\frac{\pi(X - Vt - (2R+S)d)}{2R}\right) * H(R^2 - (X - Vt - (2R+S)d)^2)e^{i\Omega t} \tag{6}$$

2.3 Application of the AMDM

The AMDM can now be applied to Eq. (6) as follows

$$\phi(X) = \Phi(X) + G^{-1}\left\{\left(K_1 - \frac{\lambda}{\gamma}\right)\frac{\mathrm{d}^2\phi(X)}{\mathrm{d}x^2} - (K_0 - \lambda)\phi(X) + \sum_{d=0}^{D-1} \frac{P_0}{2R} cos^2\left(\frac{\pi(X - Vt - (2R+S)d)}{2R}\right) * H(R^2 - (X - Vt - (2R+S)d)^2)\right\} \tag{7}$$

By substituting $\phi(X) = \sum_{m=0}^{\infty} C_m X^m$, and its second derivative into Eq. (7) one can obtain

$$\phi(X) = \Phi(X) + G^{-1}\left\{\left(K_1 - \frac{\lambda}{\gamma}\right)\sum_{m=0}^{\infty}(m+1)(m+2)C_{m+2}X^m - (K_0 - \lambda)\sum_{m=0}^{\infty}C_m X^m + \sum_{d=0}^{D-1}\frac{P_0}{2R}cos^2\left(\frac{\pi(X - Vt - (2R+S)d)}{2R}\right)H(R^2 - (X - Vt - (2R+S)d)^2)\right\} \tag{8}$$

From this a recurrence relation of C_m can be obtained as follows

$$C_0 = \phi(0), \qquad C_1 = \phi'(0), \qquad C_2 = \frac{\phi''(0)}{2}, \quad C_3 = \frac{\phi'''(0)}{6} \tag{9}$$

for $m \geq 4$ as following

$$C_m = \frac{1}{m(m-1)(m-2)(m-3)}\sum_{j=0}^{m-4}\left[\left(K_1 - \frac{\lambda}{\gamma}\right)(j+1)(j+2)C_{j+2} - (K_0 - \lambda)C_j\right] \tag{10}$$

The unknown coefficients $C_m(m = 0,1,2,3)$ are found using the boundary conditions of each section of the beam and the continuity conditions between sections.

2.4 Boundary conditions

Boundary conditions for the clamped-free case at $X = 0$ and $X = 1$ are

$$\phi(0) = \frac{\mathrm{d}\phi(0)}{\mathrm{d}X} = 0, \qquad \frac{\mathrm{d}^3\phi(1)}{\mathrm{d}X^3} = 0 \tag{11}$$

They can be described in terms of rotational and translational flexible ends to make it convenient to use with the AMDM as shown on Fig. 2.

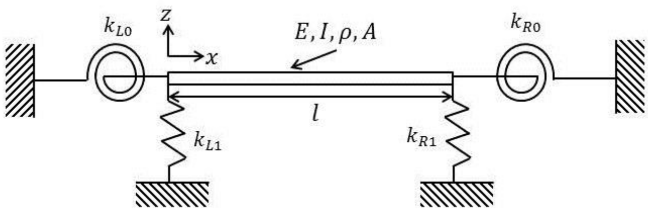

Fig. 2 Boundary condition described by rotational and translational flexible ends.

The boundary conditions are turned into dimensionless form as

$$\frac{\mathrm{d}^2\phi(0)}{\mathrm{d}X^2} - \kappa_{L0}\frac{\mathrm{d}\phi(0)}{\mathrm{d}X} = 0, \quad \frac{\mathrm{d}^3\phi(0)}{\mathrm{d}X^3} + \kappa_{L1}\phi(0) = 0$$
$$\frac{\mathrm{d}^2\phi(1)}{\mathrm{d}X^2} + \kappa_{R0}\frac{\mathrm{d}\phi(1)}{\mathrm{d}X} = 0, \quad \frac{\mathrm{d}^3\phi(1)}{\mathrm{d}X^3} - \kappa_{R1}\phi(1) = 0 \tag{12}$$

where the coefficients are made non-dimensional using

$$\kappa_{L1} = \frac{k_{L1}l^3}{EI}, \quad \kappa_{R1} = \frac{k_{R1}l^3}{EI}, \quad \kappa_{L0} = \frac{k_{L0}l}{EI}, \quad \kappa_{R0} = \frac{k_{R0}l}{EI}$$

2.5 Solution algorithm

Terms $\phi^{[1]}(X) = C_0$, $\phi^{[2]}(X) = \phi^{[1]}(X) + C_1X$, $\phi^{[3]}(X) = \phi^{[2]}(X) + C_2X^2$, $\phi^{[4]}(X) = \phi^3(X) + C_3X^3$ serve as approximate solutions with increasing accuracy as $n \to \infty$ (Adair 2019). The four coefficients $C_m(m = 0,1,2,3)$ in Eq. (9) depend on the boundary conditions of Eq. (12). The two coefficients C_0 and C_1 are chosen as arbitrary constants, and the other two coefficients C_2 and C_3 are expressed as functions of C_0 and C_1. Thus from Eq. (12) and Eq. (9) one can obtain

$$C_2 = \frac{\kappa_{L0}}{2}C_1, \quad C_3 = -\frac{\kappa_{Ro}}{6}C_0 \tag{13}$$

Thus the initial term $\Phi(X)$ can be represented as a function of C_0 and C_1 and from the Eq. (3.19) the coefficients $C_m(m \geq 4)$ are functions of C_0, C_1 and λ. By substituting $\phi^{[n]}(X)$ into the boundary conditions of Eq. (12) when $X = 1$, following is obtained

$$f_{r0}^{[n]}(\lambda)C_0 + f_{r1}^{[n]}(\lambda)C_1 = 0, \quad r = 1,2 \tag{14}$$

For non-trivial solutions, C_0 and $C_1$the frequency equation is expressed as

$$\begin{vmatrix} f_{10}^{[n]}(\lambda) & f_{11}^{[n]}(\lambda) \\ f_{20}^{[n]}(\lambda) & f_{21}^{[n]}(\lambda) \end{vmatrix} = 0 \tag{15}$$

The i^{th} estimated eigenvalue $\lambda_{(i)}^{[n]}$ corresponding to m is obtained from Eq. (15), i.e., the i^{th} estimated dimensionless natural frequency $\Omega_{n(i)}^{[n]} = \sqrt{\lambda_{(i)}^{[n]}}$ is also obtained and n is determined by

$$\left|\Omega_{n(i)}^{[n]} - \Omega_{n(i)}^{[n-1]} \leq \varepsilon\right| \tag{16}$$

where $\Omega_{n(i)}^{[n-1]}$ is the i^{th} estimated dimensionless natural frequency corresponding to $n - 1$, and ε is a pre-set sufficiently small value.

Vibration response of the Euler-Bernoulli beam resting on the Pasternak foundation is now calculated using clamped-free (cantilever) boundary conditions where the spring constants becoming, $\kappa_{L0} \to \infty, \kappa_{R0} \to 0, \kappa_{L1} \to \infty, \kappa_{R1} \to 0$ as per Eqs. (11) and (12).

The algebraic equations arising from boundary conditions in Eq. (11) and (12) with $X = 0$ and $X = 1$ are

$$\begin{aligned} &\sum_{m=0}^{n-3}(m+1)(m+2)C_{m+2} + \kappa_{R0}\sum_{m=0}^{n-2}(m+1)C_{m+1} = f_{11}^{[n]}(\lambda)C_0 + f_{12}^{[n\}}(\lambda)C_1 = 0 \\ &\sum_{m=0}^{n-4}(m+1)(m+2)(m+3)C_{m+3} - \kappa_{R1}\sum_{m=0}^{n-1}C_m = f_{21}^{[n]}(\lambda)C_0 + f_{22}^{[n]}(\lambda)C_1 = 0 \end{aligned} \qquad (17)$$

Eq. (10) will be used to define the coefficients for solving the Eq. (17).

3. RESULTS

The numerical results of frequencies of first three modes of a beam vibration are provided in Table 1 for different shear parameters, K_1 while keeping stiffness, K_0 constant. The effect of the stiffness parameter, K_0 on beam vibration frequency, with constant shear parameter was also analysed and the numerical results are provided in Table 2. From the last columns for large parameter in Table 1 and Table 2 in can be seen that the frequency is influenced the most in Table 1. From this, it can be concluded that the shear parameter has greater effect for the beam vibration frequency compared to the stiffness parameter.

Table 1 Frequencies of beam on an elastic foundation for different K_1 with $K_0 = 1$

Frequencies	$K_1 = 0$	$K_1 = 5$	$K_1 = 50$	$K_1 = 1000$
$\Omega_{n(1)}$	2.3721	3.1009	5.3574	27.942
$\Omega_{n(2)}$	5.58	5.9769	8.8382	31.061
$\Omega_{n(3)}$	7.7385	8.0402	10.388	35.199

Table 2 Frequencies of beam on an elastic foundation for different K_0 with $K_1 = 1$

Frequencies	$K_0 = 0$	$K_0 = 5$	$K_0 = 50$	$K_0 = 1000$
$\Omega_{n(1)}$	2.4954	2.6859	3.7419	5.978
$\Omega_{n(2)}$	5.654	5.6921	6.1553	7.9019
$\Omega_{n(3)}$	7.7966	7.8125	8.0045	11.315

Fig. 3 and Fig. 4 show how the frequency for each phase is affected by different foundation parameters under moving loading. Previously mentioned greater effect of shear over stiffness can also be observed here. Even though comparison was performed only until $K_0 = 15$ and $K_1 = 15$, it can be seen that the overall ratio of frequencies of any mode in Fig. 3, where shear parameter is varied keeping stiffness constant, is greater than in Fig. 4, where the stiffness is varied keeping the shear constant. Moreover, it can be noticed from both figures that the first mode is the most affected by any parameter than other modes.

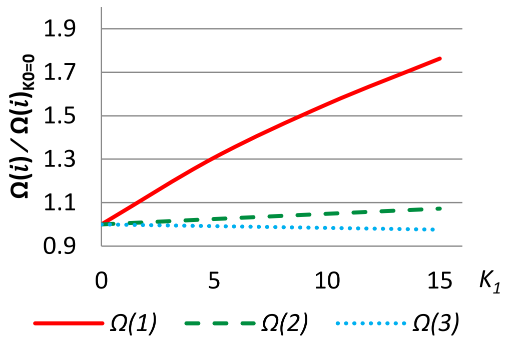

Fig. 3 The effect of the shear parameter K_1 to frequencies, at constant stiffness $K_0 = 1$.

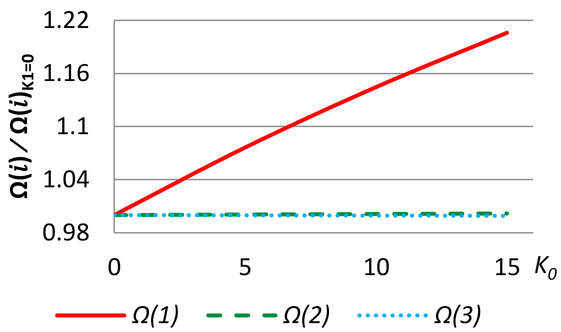

Fig. 4 The effect of the stiffness parameter K_0 to frequencies, at constant shear $K_1 = 1$.

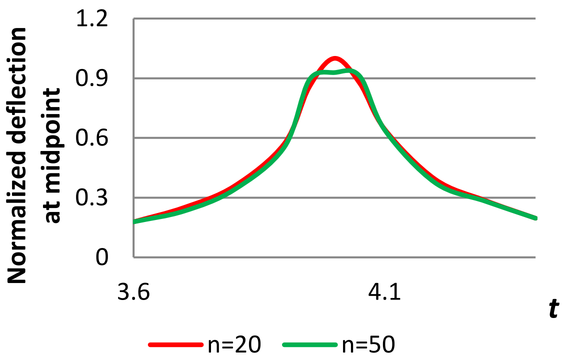

Fig. 5 The effects on the vertical deflection of the beam's midpoint by a point force when $K_0 = 0$.

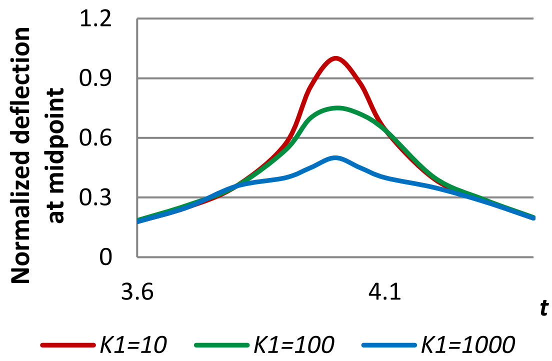

Fig. 6 The effects on the vertical deflection of the beam's midpoint by a point force for different K_1

The effect of a moving loading at the midpoint of the beam is investigated. Initially the foundation parameters were taken as $K_0 = 0$ and $K_1 = 0$, and the deflection at the midpoint is calculated for time, t. Time is given by $t = l/2v$, where l is the length of the beam, v is the velocity of the moving loading. Fig. 5 illustrates the deflection when $v = 50m/s$ for different iterations.

The effect of the foundation shear parameter on the deflection of the beam at the middle is also analyzed. The results are presented in Fig. 6. From Fig. 6 it can be seen that an increase in the shear parameter decreases the amplitude of deflection. It can be concluded that the shear parameter has a significant effect on preventing the deflection of the beam under the moving load.

4. CONCLUSIONS

An analytical solution method, namely the Adomian Modified Decomposition method was developed that gives efficient, fast and accurate results for the fourth-order nonlinear differential equation which describes the transverse vibrations of an Euler-Bernoulli beam resting on a two parameter foundation. The developed AMDM based solution algorithm determines the free and forced vibration frequencies of a beam and it is relatively easy to apply boundary conditions. Quick convergences with accurate results were observed, especially for the first vibration mode. Analysis shows that the foundation shear parameter has a greater effect on vibration frequency of a beam compared to the stiffness parameter. Moreover, investigation of the beam under harmonic moving loading showed that the shear parameter had a significant effect on decreasing the deflection of the beam under moving loading.

ACKNOWLEDGMENTS

This investigation was supported by Nazarbayev University Small Competitive Grant no. 090118FD5317 and ORAU Grant no. SOE2017003.

REFERENCES

Adair, D., Ibrayev, A., Tazabekova, A. and Kim, J. (2019), "Free Vibrations with Large Amplitude of Axially Loaded Beams on an Elastic Foundation Using the Adomian Modified Decomposition Method." *Shock and Vibration*, Vol. 2019.

Adair, D., Ibrayev, A., Kim, J. and Jaeger, M., (2019), "Application of the Adomian Modified Decomposition Method for a Beam with a Single Section Discontinuity Resting on an Elastic Foundation," *26th International Congress on Sound and Vibration*, Montréal, QC, Canada

Adomian, G. (1994), *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer Academic Publishers, Boston, MA, USA.

De Rosa, M. A. and Maurizi, M. J. (1998), "The influence of concentrated masses and Pasternak soil on the free vibrations of Euler beams - exact solution." *J. Sound and Vibration*, Vol. 212, 573-581

Taha, MH (2017), "General analysis of stressed beams on elastic foundation subjected to discontinuous loads." *Ships and Offshore Structures*, Vol. 12(3), 388-394

Tazabekova A., Adair D., Ibrayev A., Kim J. (2018), "Free Vibration Calculations of an Euler-Bernoulli Beam on an Elastic Foundation Using He's Variational Iteration Method," *Proceedings of China-Europe Conference on Geotechnical Engineering. Springer Series in Geomechanics and Geoengineering,* Vienna, Austria

Wazwaz, A.M. (1999), "A reliable modification of Adomian's decomposition method." *Appl. Math. Comput*., Vol. 102, 77-86

Wazwaz., A.M. and El-Sayed, S.M. (2001), "A new modification of the Adomian decomposition method for linear and nonlinear operators." *Appl. Math. Comput*., Vol. 122 (3), 393-405