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Orthotropic yield criteria based on crystallographic orientations and their effect on the development of plastic zones in thin discs

Yaroslav A. Erisov[1)] and *Sergei V. Surudin[2)]

[1), 2)] *Samara State Aerospace University, Samara 443086, Russia*
[1)] *yaroslav.erisov@mail.ru*
[2)] *innosam63@gmail.com*

ABSTRACT

Using the yield criteria proposed the effect of plastic anisotropy on the development of a plastic zone in a thin annular disc subject to internal pressure is revealed. It is shown that, in contrast to isotropic material, the plastic zone may start to develop from the outer surface. In this case the solution is purely analytic. If the plastic zone starts to develop from the inner surface of the disc then a numerical technique is in general necessary. This technique is based on the method of characteristics.

1. INTRODUCTION

Modern stage of technological development in aircraft, automotive, shipbuilding and other industries is marked by continuous updating, searching and designing of new advanced manufacturing methods especially metal forming processes. Creation of efficient and scientifically founded metal forming technologies is associated with necessity for detailed study of material properties and fullest usage of these properties in engineering design.

The most specific property, resided in majority of real materials, is anisotropy, which is the cause of crystalline structure and texture formation during plastic flow of metal. Neglect of this fundamental material characterization in process design not only reduces potential deformational capabilities of blank, but also leads to other undesirable phenomena: increased metal usage, limitation of ultimate strain, size and shape distortion, performance worsening, etc. On the other hand, efficient anisotropy is a serious intensification factor for metal forming processes and increasing performance of parts in certain directions.

In the recent times it has been putted more emphasis on theoretical and experimental research of plastic behavior of anisotropic bodies. However, there are some unsolved problems, which are associated with the further development of anisotropic medium plasticity in the form, appropriate in engineering and process design.

For example, widely used at the moment yield functions allow performing computations of plastic behavior of anisotropic blanks with certain accuracy and complexity. Nevertheless, these criteria do not allow solving inverse problem – defining effective for forming and performance crystallographic orientations, which are the cause of anisotropy.

In consideration of the foregoing, in this work because of energy approach it is deduced the yield criteria, which describes the influence of crystalline anisotropy and anisotropy, induced by crystallographic texture, on elastic-plastic transition of orthotropic material with cubic lattice.

2. PROPOSED YIELD CRITERIA

Typically, in calculations of metal forming processes it is used von Mises criterion (Hosford 2005). According to it yielding occurs when the elastic distortional strain energy U_{dis} reaches a critical value U_{dis}^{yield} :

$$F = U_{dis} - U_{dis}^{yield} = 0 .$$

Express the distortional strain energy U_{dis} as difference between full elastic strain energy U_{full} and strain energy due to change of volume U_{vol}:

$$U_{dis} = U_{full} - U_{vol} , \qquad (1a)$$

or considering that strain energy is equal to half of dot product of stress and strain tensors (in case of volume strain energy – spherical tensors), receive

$$U_{dis} = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} - \frac{1}{6}\sigma_{ii}\varepsilon_{jj} , \qquad (1b)$$

where σ_{ij} and ε_{ij} are stress and strain tensors.

Express the strain tensor in Eq. (1b) using the Hooke's law (Hosford 2005):

$$\varepsilon_{ij} = S_{ijkl}\sigma_{ij} , \qquad (2)$$

where S_{ijkl} is compliance tensor in principal axes of anisotropy (for rolled sheet: axis 1 is along rolling direction; axis 2 is transverse direction and axis 3 is thickness direction).

In turn, express components of tensor S_{ijkl} through compliance tensor S'_{pqrs}, which is associated with crystallographic axes [001], [010] and [100] of crystal lattice. Considering that for orthotropic body with cubic crystal lattice, such tensor contains only three independent components, receive (Adamesku 1985):

$$\begin{aligned} S_{iiii} &= S'_{1111} - 4S'_{2323}\left(A'-1\right)\Delta_i, \\ S_{iijj} &= S'_{1122} - 2S'_{2323}\left(A'-1\right)\left(\Delta_k - \Delta_i - \Delta_j\right), \\ S_{ijij} &= S'_{2323} - 2S'_{2323}\left(A'-1\right)\left(\Delta_k - \Delta_i - \Delta_j\right), \end{aligned} \qquad (3)$$

where A' is anisotropy parameter of crystal lattice:

$$A' = \frac{S'_{1111} - S'_{1122}}{2S'_{2323}};$$

Δ_i are texture parameters, which for certain crystallographic orientation $\{hkl\}\langle uvw\rangle$ are defined as

$$\Delta_i = \frac{h_i^2 k_i^2 + k_i^2 l_i^2 + l_i^2 h_i^2}{\left(h_i^2 + k_i^2 + l_i^2\right)^2};$$

h_i, k_i, l_i are Miller indices, defining *i*-th direction in crystal about principal axes of anisotropy.

Assuming that Hooke's law applies till yielding, and substituting Eq. (2) and Eq. (3) into Eq. (1b) result in

$$U_{dis} = \frac{S'_{2323}}{15}\left(3 + 2A'\right) K_{ijkl}\sigma_{ij}\sigma_{kl}, \tag{4}$$

where K_{ijkl} is material tensor:

$$K_{ijkl} = \begin{bmatrix} \eta_{12}+\eta_{31} & -\eta_{12} & -\eta_{31} & 0 & 0 & 0 \\ -\eta_{12} & \eta_{12}+\eta_{23} & -\eta_{23} & 0 & 0 & 0 \\ -\eta_{31} & -\eta_{23} & \eta_{23}+\eta_{31} & 0 & 0 & 0 \\ 0 & 0 & 0 & 4\left(\frac{5}{2}-\eta_{12}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 4\left(\frac{5}{2}-\eta_{12}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 4\left(\frac{5}{2}-\eta_{12}\right) \end{bmatrix};$$

η_{ij} are generalized anisotropy parameters:

$$\eta_{ij} = 1 - \frac{15\left(A'-1\right)}{3+2A'}\left(\Delta_i + \Delta_j - \Delta_k - \frac{1}{5}\right), \tag{5}$$

It follows from Eq. (5) that anisotropy is determined by anisotropy of crystal lattice A' (alloy grade) and crystallographic texture Δ_i (treatment). Here elastically isotropic material is also isotropic in plastic region.

For example, for aluminum alloy components of compliance tensor S'_{pqrs} are $S'_{1111} = 15.80$ TPa^{-1}; $S'_{1122} = -5.80$ TPa^{-1} and $S'_{2323} = 8.95$ TPa^{-1} (Hellwege 1966) then $A' = 1.207$. Substituting this value into Eq. (5) and assuming shear rolling texture {001}<110> (for which $\Delta_1 = 0.25$, $\Delta_2 = 0.25$ and $\Delta_3 = 0$) result in the following generalized anisotropy parameters $\eta_{12} = 0.828$, $\eta_{23} = 1.115$ and $\eta_{31} = 1.115$. Or assuming Goss recrystallization texture {011}<001> (for which $\Delta_1 = 0$, $\Delta_2 = 0.25$ and $\Delta_3 = 0.25$) results in $\eta_{12} = 1.115$, $\eta_{23} = 0.828$ and $\eta_{31} = 1.115$.

In order that constitutive equations are revealed invariant, it is necessary to accept that distortion strain energy is equal to according energy of isotropic medium:

$$U_{dis}^{iso} = \frac{2}{15} S'_{2323} \left(3 + 2A'\right) \sigma_i^2 , \tag{6}$$

where σ_i is effective stress.

Equating right parts of Eq. (4) and Eq. (6) gives

$$\sigma_i = \frac{1}{\sqrt{2}} \sqrt{K_{ijkl} \sigma_{ij} \sigma_{kl}} \tag{7a}$$

or in expanded form

$$\begin{aligned} \sigma_i = \frac{1}{\sqrt{2}} \Bigg\{ & \eta_{12} \left(\sigma_{11} - \sigma_{22}\right)^2 + \eta_{23} \left(\sigma_{22} - \sigma_{33}\right)^2 + \eta_{31} \left(\sigma_{33} - \sigma_{11}\right)^2 + \\ & + 4 \left[\left(\frac{5}{2} - \eta_{12}\right) \sigma_{12}^2 + \left(\frac{5}{2} - \eta_{23}\right) \sigma_{23}^2 + \left(\frac{5}{2} - \eta_{31}\right) \sigma_{31}^2 \right] \Bigg\}^{1/2} \end{aligned} \tag{7b}$$

In case of isotropic medium, when $A' = 1$ or $\Delta_i = 1/5$ [2], Eq. (7b) becomes the form of von Mises criterion.

The relations between plastic strains and the stress state are expressed as (Hill 1950):

$$d\varepsilon_{ij} = d\lambda \frac{\partial \sigma_i}{\partial \sigma_{ij}} , \tag{8}$$

where $d\lambda$ is Lagrange multiplier.

Assuming that at small elastic-plastic strains anisotropy does not change ($\eta_{ij} = const$), and differentiating Eq. (7a) with respect to Eq. (8) result in

$$
\begin{aligned}
d\varepsilon_{11} &= \frac{1}{2}\frac{d\lambda}{\sigma_i}\left[\eta_{12}\left(\sigma_{11}-\sigma_{22}\right)-\eta_{31}\left(\sigma_{33}-\sigma_{11}\right)\right] \\
d\varepsilon_{22} &= \frac{1}{2}\frac{d\lambda}{\sigma_i}\left[\eta_{23}\left(\sigma_{22}-\sigma_{33}\right)-\eta_{12}\left(\sigma_{11}-\sigma_{22}\right)\right] \\
d\varepsilon_{33} &= \frac{1}{2}\frac{d\lambda}{\sigma_i}\left[\eta_{31}\left(\sigma_{33}-\sigma_{11}\right)-\eta_{23}\left(\sigma_{22}-\sigma_{33}\right)\right] \\
d\varepsilon_{12} &= 2\frac{d\lambda}{\sigma_i}\left(\frac{5}{2}-\eta_{12}\right)\sigma_{12} \\
d\varepsilon_{23} &= 2\frac{d\lambda}{\sigma_i}\left(\frac{5}{2}-\eta_{23}\right)\sigma_{23} \\
d\varepsilon_{31} &= 2\frac{d\lambda}{\sigma_i}\left(\frac{5}{2}-\eta_{31}\right)\sigma_{31}
\end{aligned}
\tag{9}
$$

Solving Eq. (9) with additional condition

$$
\left(\sigma_{11}-\sigma_{22}\right)+\left(\sigma_{22}-\sigma_{33}\right)+\left(\sigma_{33}-\sigma_{11}\right)=0\,,
$$

receive conversed expressions:

$$
\begin{aligned}
\sigma_{11}-\sigma_{22} &= 2\frac{\sigma_i}{d\lambda}\frac{1}{\xi\eta_{12}}\left(\frac{d\varepsilon_{11}}{\eta_{31}}-\frac{d\varepsilon_{22}}{\eta_{23}}\right) \\
\sigma_{22}-\sigma_{33} &= 2\frac{\sigma_i}{d\lambda}\frac{1}{\xi\eta_{23}}\left(\frac{d\varepsilon_{22}}{\eta_{12}}-\frac{d\varepsilon_{33}}{\eta_{31}}\right) \\
\sigma_{33}-\sigma_{11} &= 2\frac{\sigma_i}{d\lambda}\frac{1}{\xi\eta_{31}}\left(\frac{d\varepsilon_{33}}{\eta_{23}}-\frac{d\varepsilon_{11}}{\eta_{12}}\right) \\
\sigma_{12} &= \frac{1}{2}\frac{\sigma_i}{d\lambda}\frac{d\varepsilon_{12}}{\frac{5}{2}-\eta_{12}};\ \sigma_{23}=\frac{1}{2}\frac{\sigma_i}{d\lambda}\frac{d\varepsilon_{23}}{\frac{5}{2}-\eta_{23}}; \\
\sigma_{31} &= \frac{1}{2}\frac{\sigma_i}{d\lambda}\frac{d\varepsilon_{31}}{\frac{5}{2}-\eta_{31}};\ \xi=\frac{1}{\eta_{12}}+\frac{1}{\eta_{23}}+\frac{1}{\eta_{31}}
\end{aligned}
\tag{10}
$$

Substituting Eq. (10) into Eq. (7b) defines effective strain $d\varepsilon_i$:

$$d\varepsilon_i = d\lambda = \sqrt{2}\left\{\frac{1}{\xi^2}\left[\frac{1}{\eta_{12}}\left(\frac{d\varepsilon_{11}}{\eta_{31}} - \frac{d\varepsilon_{22}}{\eta_{23}}\right)^2 + \frac{1}{\eta_{23}}\left(\frac{d\varepsilon_{22}}{\eta_{12}} - \frac{d\varepsilon_{33}}{\eta_{31}}\right)^2 + \right.\right.$$

$$\left.\left. + \frac{1}{\eta_{31}}\left(\frac{d\varepsilon_{33}}{\eta_{23}} - \frac{d\varepsilon_{11}}{\eta_{12}}\right)^2\right] + \frac{1}{4}\left[\frac{d\varepsilon_{12}^2}{\frac{5}{2}-\eta_{12}} + \frac{d\varepsilon_{23}^2}{\frac{5}{2}-\eta_{23}} + \frac{d\varepsilon_{31}^2}{\frac{5}{2}-\eta_{31}}\right]\right\}^{1/2} \tag{11}$$

3. CONCLUSIONS

Based on energy approach it is deduced the yield criteria, which describes the influence of crystalline anisotropy and anisotropy, induced by crystallographic texture, on elastic-plastic transition of orthotropic material with cubic lattice. It is shown that anisotropy is determined by anisotropy of crystal lattice, i.e. alloy grade, and crystallographic texture, i.e. treatment. Using the yield criteria proposed the effect of plastic anisotropy on the development of a plastic zone in a thin annular disc subject to internal pressure is revealed.

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