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Exact Solutions to the Oblique Shock Wave Equation

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ABSTRACT

The purpose of the paper is to review and analyze analytical, approximate and iterative solutions and characteristics of oblique shock wave equation in a supersonic freestream. The closed-form solutions of the cubic polynomial equation are having real and conjugate complex roots and associates with strong and weak shock wave. A computer program is written to compute the angle of the oblique shock wave. The exact solution of the oblique shock equation in terms of tanβ is found to be most convenient and efficient way to compute the oblique shock wave angle β for given upstream flow conditions. The numerical algorithm is based on the closed-form solution that can easily use for rapid calculations of the oblique shock wave angle. The subroutine will be useful for design of an air-breathing hypersonic vehicle.

Introduction

The oblique shock wave theory essentially requires for designing a wave rider configuration and an air-breathing hypersonic vehicle (Bolender and Doman, 2007). The oblique shock wave angle β is needed as an explicit function of the upstream Mach number M_1 and the flow deflection angle θ as depicted in Figure 1. It is commonly known as θ-β-M relationship of the oblique shock wave theory associated with compressible gas-dynamics. Ames Research Staff (1953) has mentioned in their report that the analytical solution of the oblique shock equation cannot be arrived and tabulated values of θ-β-M.

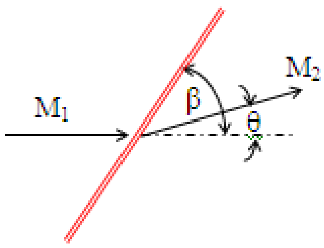

Figure 1 Flow through an oblique shock wave

Thompson (1950) has derived a cubic polynomial equation in term of sinβ of the oblique shock wave equation. Briggs (1963) and Mascitti (1969) have obtained analytical solution of the cubic equation. Naylor (1954) has presented a solution of the shock-wave cubic equation that allows computation of the oblique shock wave angle without tables. Hartley et al. (1991) have carried out real-time application of the exact and approximate solutions to the oblique shock wave equations.

The cubic polynomial equation of the oblique shock wave equation in terms of tanβ had been derived by Wellmann (1972) and the analytical solutions were published by Wolf (1993). An analytical solution was also obtained by Emanuel (2000) and compared with the tabulated value of β by Anderson (2004). Bar-Meir (2013) has discussed in detail the characteristics of the real and complex conjugate roots of the cubic polynomial equation of the oblique shock wave.

Duo et al. (1992), Powers (1992) and Agnone (1994) have discussed the approximate formula for weak and strong shock wave angles. Houghton and Brock (1988) and Houghton and Carpenter (2005) presented iteration method for the solution of cubic polynomial equation. Rudd and Lewis (1998) have compared the closed-form solutions with the iterative scheme (Houghton and Brock, 1988). They concluded that the computer algorithm of the iterative method is too complicated and consuming more computer time as compared to the analytical solution.

The above literature survey reveals that analytical, approximate and iterative methods are available to obtain the value of the oblique shock wave angle for the given upstream flow conditions. The present paper presents the closed-form solution for the shock wave angle based on the oblique shock wave theory. The roots are obtained using the Cardan cubic polynomial equation (Grewal, 1993). A computer program is developed based on the exact solution of the oblique shock theory which can easily use in the design phase of an air-breathing hypersonic vehicle.

Analytical Solution

The relationship between the oblique shock wave angle β, the flow deflection angle θ across the oblique shock wave and upstream Mach number M_1 (Liepmann and Roshko, 2007) is

$$\frac{tan(\beta-\theta)}{tan\beta}=\frac{(\gamma-1)M_1^2\,sin^2\beta+1}{(\gamma+1)M_1^2\,sin^2\beta} \tag{1}$$

where γ is the ratio of specific heats and subscript 1 represents upstream supersonic Mach number. The above equation shows an implicit relation between θ-β-M. Eqn (1) becomes zero at $\beta = \pi/2$ and at $\beta = \alpha = \sin^{-1}(1/M_1)$, where α denotes Mach wave angle.

Within this range β is positive and must therefore have a maximum value of flow deflection angle θ_{max}. There are two ways to express the polynomial equation of Eqn (1). Thompson (1950) has obtained following expression for Eqn (1) as

$$sin^6\beta + b\,sin^4\beta + c\,sin^2\beta + d = 0 \tag{2}$$

where

$$b = -\left[\frac{M_1^2+2}{M_1^2} + \gamma\,sin^2\theta\right]$$

$$c = \frac{2M_1^2+1}{M_1^4} + \left[\frac{(\gamma+1)^2}{4} + \frac{\gamma-1}{M_1^2}\right]sin^2\theta$$

$$d = -\frac{cos^2\theta}{M_1^4}$$

The Cardan equations give three roots for $sin^2\beta$ from the foregoing equation. The direct computation of oblique shock wave properties with freestream Mach number and flow deflection angle as the independent variables is used to determine the strong and the weak shock wave angle. One of the root of the cubic equation is real and the solutions of Eqn (2) may written as

$$\beta_s = tan^{-1}\left(\sqrt{\frac{x_s}{1-x_s}}\right) \tag{3a}$$

$$\beta_w = tan^{-1}\left(\sqrt{\frac{x_w}{1-x_w}}\right) \tag{3b}$$

where

$$v = \frac{(3d-b^2)}{9}, \qquad w = \frac{(9bc-27d-2b^2)}{54}, \qquad D = v^3 + w^2$$

$$x_s = -\frac{b}{3} + 2\sqrt{-v}\,cos\phi, \qquad x_w = -\frac{b}{3} - \sqrt{-v}\left(cos\phi - \sqrt{3}\,sin\phi\right), \qquad \phi = \frac{1}{3}\left(tan^{-1}\frac{\sqrt{-D}}{w} + \Delta\right)$$

where subscripts s and w represent the strong and the weak shock, respectively. If $\Delta = 0$ then $w \geq 0$; and if $\Delta = \pi$, then $w < 0$. Normal shock wave occurs just at $\theta = 0$ and $\beta = \pi$. If $\theta > \theta_{max}$, then no solution exists for a straight oblique shock wave. If $\theta < \theta_{max}$, then there are two values of β for a given value of M_1. The large value gives a strong shock solution where downstream M_2 is subsonic. The small value gives the weak shock solution where M_2 is supersonic except for a small region near θ_{max}. In nature, the weak shock solution is favored and that one usually occurs.

By writing the another general form of Eqn. (1) in a cubic relation for tanβ, Wellmann (1972) derived following equation

$$tan\theta\left[\left(\frac{\gamma-1}{2}\right)+\left(\frac{\gamma+1}{2}\right)tan^2\alpha\right]tan^3\beta-tan^2\beta+tan\theta\left[\left(\frac{\gamma+1}{2}\right)+\left(\frac{\gamma+3}{2}\right)tan^2\alpha\right]tan\beta+tan^2\alpha=0 \quad (4)$$

Let us introduce a new variable $tan\beta=y$ and Eqn (4) becomes

$$y^3+by^2+cy+d=0 \quad (5)$$

where

$$b=\left(\frac{\frac{2}{\gamma-1}\frac{cos\theta}{sin\theta}}{M_1^2+\frac{2}{\gamma-1}}\right),\ c=\left(\frac{\frac{\gamma+1}{\gamma-1}M_1^2+\frac{2}{\gamma-1}}{M_1^2+\frac{2}{\gamma-1}}\right),\ d=\left(\frac{\frac{2}{\gamma-1}\frac{cos\theta}{sin\theta}}{M_1^2+\frac{2}{\gamma-1}}\right)\left(1-M_1^2\right)$$

Let us introduce another variable say $y=x-b/3$ in Eqn (5) and reduced to

$$x^3+vx+w=0 \quad (6)$$

The three roots of Eqn (6) are as following

$$x_1=A+B \quad (7a)$$

$$x_{2,3}=-\frac{1}{2}(A+B)\pm\frac{\sqrt{3}}{2}(A-B)i \quad (7b)$$

where

$$A=\left[-\frac{w}{2}+(D)^{1/2}\right]^{1/3},\qquad B=\left[-\frac{w}{2}-(D)^{1/2}\right]^{1/3},\ \text{and}\ D=\frac{w^2}{4}+\frac{v^3}{27}$$

If D > 0, then one real root of Eqn (5) is

$$\beta=tan^{-1}\left\{\left(A+B\right)-\frac{b}{3}\right\} \quad (8a)$$

and the other roots depend on the magnitude of D. They can be expressed as

$$\beta_1 = tan^{-1}\left\{-2B_1 \cdot cos\left[cos^{-1}\left(\frac{\phi}{3}\right)\right] - b\right\} \tag{8b}$$

$$\beta_2 = tan^{-1}\left\{-2B_1 \cdot cos\left[cos^{-1}\left(\frac{\phi}{3}\right) + \frac{\pi}{2}\right] - b\right\} \tag{8c}$$

$$\beta_3 = tan^{-1}\left\{-2B_1 \cdot cos\left[cos^{-1}\left(\frac{\phi}{3}\right) - \frac{\pi}{2}\right] - b\right\} \tag{8d}$$

where

$$B_1 = \sqrt{-\frac{v}{3}}, \qquad A_1 = -\frac{w}{2}\frac{1}{B_1^3} \quad \text{and} \qquad \phi = cos^{-1}(A_1)$$

A numerical algorithm is written based on the magnitude of D. The computer subroutine will compute the real root, the weak and the strong shock wave angle with the specific upstream Mach number and the flow deflection angle.

Iterative Solution

An iterative solution (Rajaraman 1996) for the cubic equation can also be used to calculate β. For any root $(\tan\beta)_k$ is written with respect to the coefficient of the polynomial of Eqn (5) as

$$|y|_k \leq \sqrt{b^2 - 2a} \tag{9}$$

where subscript *k* is the root of Eqn (5). This gives an upper bound for all the roots of the polynomial equation. All the coefficients of the polynomial equation are real. Then all the roots of the equation are real or the pairs of roots are complex conjugates. The Newton-Raphson method is simple to calculate the real root of the cubic equation. The iteration yields the real root $y_1 = \tan\beta$ at k = 1. Eqn (5) reduced the following quadratic equation

$$y^2 + (b - y_1)y + (d - y_1) = 0 \tag{10}$$

Solution of Eqn (10) can be obtained using following formula

$$y = -(b - y_1) \pm \frac{\sqrt{(b - y_1)^2 - 4(d - y_1)}}{4} \tag{11}$$

The real root can be obtained by solving Eqn (5). The iteration scheme can be initiated with an initial value using Eqn (9). Other roots can be calculated by solving Eqn (11).

The iterative scheme requires more computer time to obtain the oblique shock wave angle for the specific flow conditions and deflection angle.

Conclusion

The paper compares the relative performance of analytical, approximate and iterative solutions of the oblique shock wave equation. The strong and weak shock wave angle can be calculated from the closed-form solution for given upstream flow conditions. The analytical solutions are useful and would lead to saving in computer time. The numerical algorithm is efficient, simple and straightforward to implement in designing air-breathing hypersonic vehicle.

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Appendix A: Computer program for oblique shock wave

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SUBROUTINE OBLIQUE (EM1, THETA, BETA, EM2, GAMA, IFL)
C       EM1 - FREESTREAM MACH NUMBER
C       THETA - FLOW DEFLECTION ANGLE (in DEGREES)
C       BETA - SHOCK ANGLE
C       GAMA - RATIO OF SPECIFIC HEATS
C       EM2 - MACH NUMBER AFTER SHOCK
C       IFL - PARAMETER = 0 FOR ATTACHED SHOCK IFL - PARAMETER = 1
C       FOR DETACHED SHOCK (FLOW PROBLEM HAS TO SOLVE SEPARATELY)
C       THETA = 90 deg FOR THE NORMAL SHOCK
        RM(X) = SQRT((GM1*X*X+2.)/(2.*G*X*X-GM1))
        EM2=EM1
        IF (EM1.LT.1.)  RETURN
        IF (THETA.EQ.0.)RETURN
        IFL=0
        PI=3.14159265
        THI=THETA*PI/180.
        G=GAMA
        GP1=G+1.
        GM1=G-1.
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```
      IF (THETA.EQ.90.) GO TO 6
      C=2./GM1*COS(TH1)/SIN(THA)/(EM1*EM1+2./GM1)
      B=(GP1/GM1*EM1*EM1)+2./GM1)/(EM1*EM1+2./GM1)
      A=C*(1.-EM1*EM1)
      P=-A*A/3.+B
      Q=2.*A*A*A/27.-A*B/3.+C
      QQ=(P/3.)**3+(Q/2.)**2
      IF (QQ.LT.0.) GO TO 1
      QQ=SQRT(QQ)
      A1=QQ-Q/2.
      B1=-QQ-Q/2.
      A1=A1/ABS(A1)*ABS(A1)**(1./3.)
      B1=B1/ABS(B1)*ABS(B1)**(1./3.)
      B1=A1+B1-A/3.
      B1=ATAN(B1)
      IF(B1.GT.0.) GO TO 5
      WRITE (7,100)THETA
100   FORMAT ('DEFLECTION',F10.6,' IS GREATER THAN MAX DEFLECTION)
      IFL=1
      RETURN
1     B1=SQRT(-P/3.)
      A1=-Q/2./B1**3
      A1=ACOS(A1)
      Z1=2.*B1*COS(A1/3.)-A/3.
      Z2=-2.*B1*COS(A1/3.+P1/3.)-A/3.
      Z3=-2.*B1*COS(A1/3.-PI/3.)-1/3.
      IF(Z2.GT.Z1)GO TO 2
```

```
      A1=Z1
      Z1=Z2
      Z2=A1
2     IF(Z3.GT.Z1) GO TO 3
      A1=Z1
      Z1=Z3
      Z3=A1
3     IF(Z3.LT.0.)WRITE(7,100)
      IF(Z3.LT.0)IFL=1
      IF(Z3.LT.0.)RETURN
      B1=ATAN(Z2)
5     BETA=B1*180./PI
      RM1=EM1*SIN(B1)
      EM2=RM(RM1)
      EM2=EM2/(SIN(B1-TH1)
      GO TO 7
6     BETA=90.
      RM1=EM1
      EM2=RM(RM1)
7     RR1=RR(RM1)
      RETURN
      END
```